The Chess Board Fable

This document is located at http://lasttechage.wordpress.com/pdf-references/

Exponential growth

This is an example of how things grow when the process is unrestrained by outside influences … A.K.A. *unconstrained*, *initiation*, or *exponential* growth. In pop culture, *exponential* seems to mean something like "Awesome!" or "Golly that's big!" It actually is a technical word for a specific process that builds slowly at first, then expands at a nearly inconceivable rate.

What it means in math

The word indicates an algebraic function of form: constant to a variable power.

Examples would be 2^x or $10^{t/T_{10}}$ or e^{t/T_e}. The last constant, e, is the irrational number 2.71828… This pops up whenever you look at a process that grows continuously (not in steps like a bank account)

The Chess Board Fable

There are many variation to this tall story. Here is one:

Ancient story:

Kingdom is ravaged by terrible beast – destroys crops, children, villagers, taxpayers.
King sends out plea for help ***Brave Young Man*** responds.

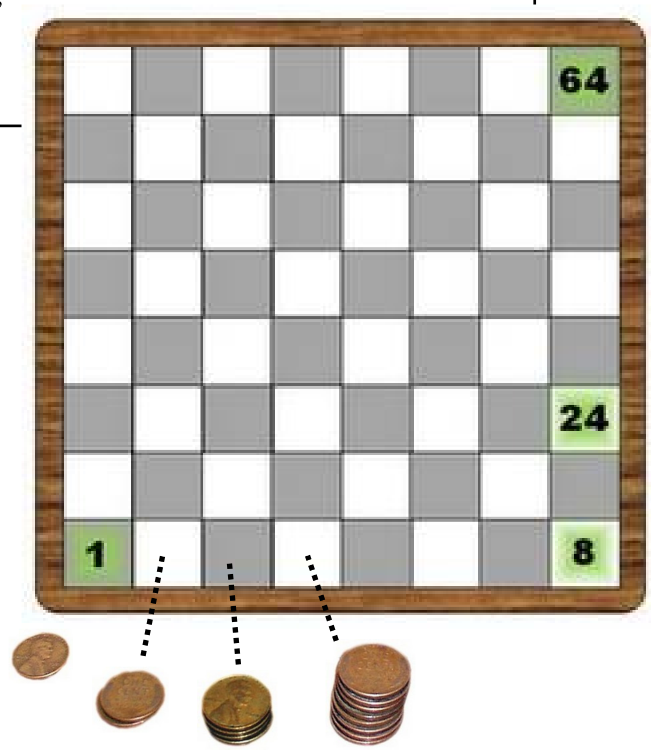
KING: Save us, I will give *anything* in reward.
BYM: Ok, my reward will be your daughter's hand in marriage.
KING: Yes, yes, anything!

Month later, ***Kingdom saved!***
Victorious **BYM returns** bloody, limping, arm in sling, ready for his new wife …

King reneges, Oh … About that daughter thing – love to oblige 'n all that. Can't.
Sorry dude. Rich old count down the way just offered a great dowry

BYM: Well, ok, I withdraw my claim if you give me the results of this…
Day 1: 1 penny on square 1 of chess board,
Day 2: 2 pennies on second square,
Day 3: 4 pennies on third,

Each successive day, 2× previous day.



King tries to think about this.

"Hmm, on the fourth day, only 15 pennies, total.
maybe Square 8… *I can count that* 128
Square 16 8 more squares 1024 (8*128)
Square 24 ≈ 8*1024 8192
Square 64 … 268,435,456 only that?

Is BYM stupid, or could I have made a mistake?"

There is a Royal Law that states the King is never wrong,
… obviously, he must be right.
The King accepts.

Is King right or wrong?

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The days pass by and the first row fills up

The stack on Square 8 is more than 7" high.

The King saw what was happening:

- Tomorrow's gain in height is always proportional to the stack's height today.

Tomorrow — the stack will be twice as tall as it is **today**.

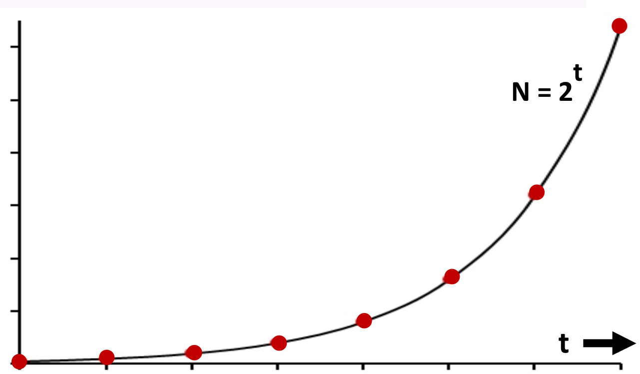
Hey! No problem. Only 7" and 128 coins.
Tomorrow, that's 256 pennies and 14"

- The royal treasury has truly *huge* resources. Tomorrow's demand will be trivial to meet, as well as the next day's*!*

"This is easy, surely we can keep going "forever."

King redid his work, this time doubling each square

Wait … Will the stack on square 16 really be 150 feet high*?*



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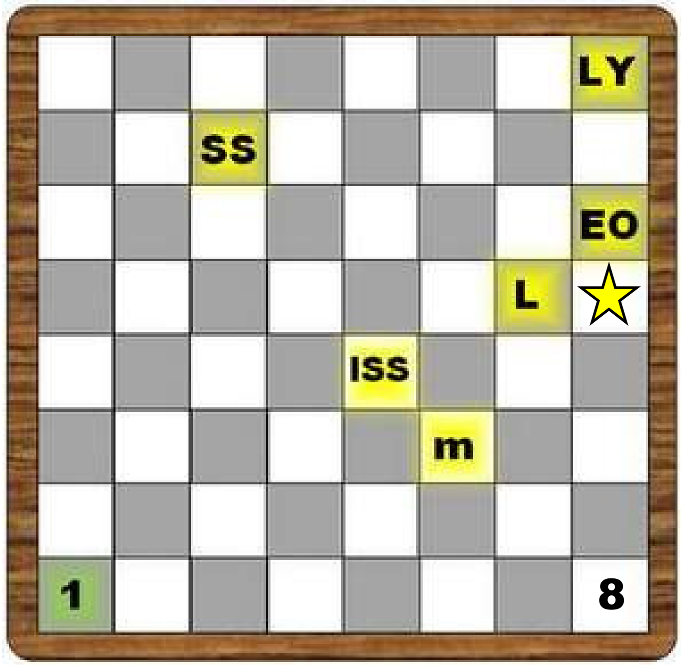
Could the new guesses be right ?

Call in ...* the *Court Mathematician

The CM pondered, measured, calculated.
hmm ... penny 1.4 mm thick (0.055"), ... doubling ...

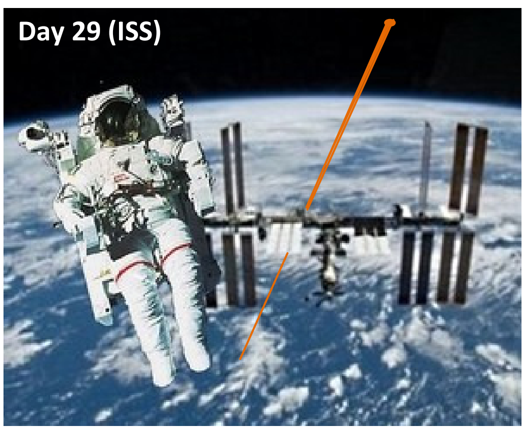
By Day 22 he was ready to explain
Here is his final diagram



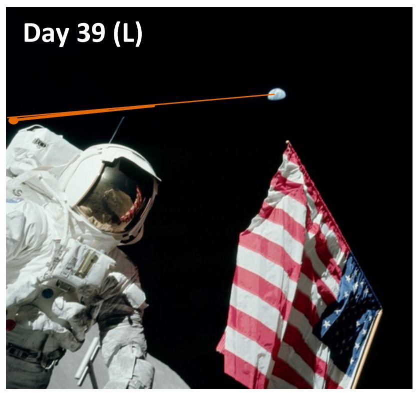
The game had reached square 22 (marked **m**), the stack towered more than **1 mile,** with 2^{21} =2,097,152 coins. Tomorrow, the stack height will be twice that.

No outside constraints to growth:

CM: "M'lord, let us <u>assume</u> that progress is ***limited only by the game rules*** and we can, build a new tower each and every day. Here is what will happen ... "

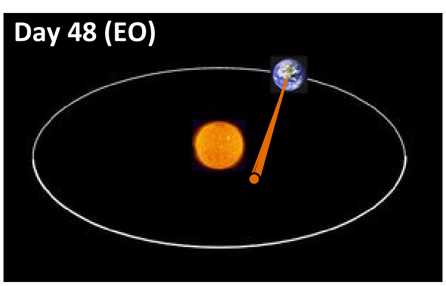


Day 29 Floating about the Station, catching a few rays in a quiet orbit. Suddenly a coppery spike shoots from Earth and zips past.
Tomorrow's stack will go twice this distance.

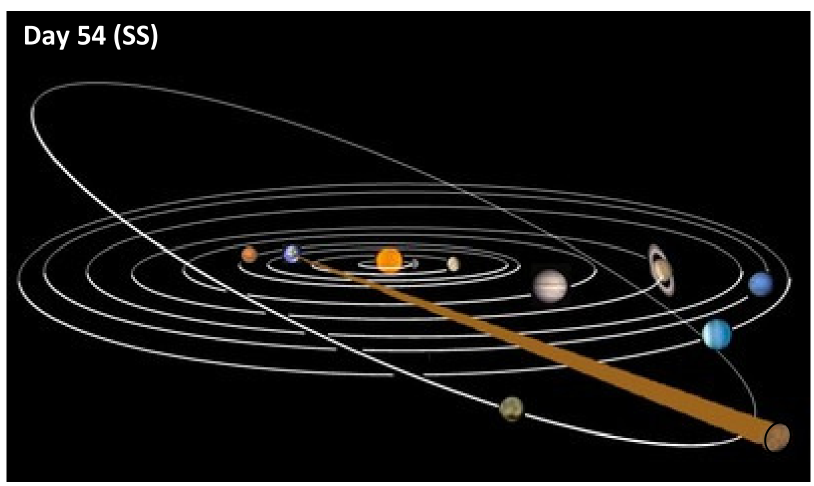


Day 39 Imagine the surprise at our Lunar Colony base as the copper spike shoots past the moon!

Day 40 The pennies in the stack out number the twinkling stars in the Milky Way (200-400×10^9)



Day 48 Length of the coin stack exceeds Earth-Sun distance (93 M mi, 150 M km) .
Tomorrow > diameter *of Earth orbit*



Day 54 Penny stack longer than the diameter of the solar system.
7.5 billion miles or 12×10^9 km
Tomorrow's stack exceeds 2 solar system widths.

Day 64 Stack exceeds 1 light year.
Day 66... at Alpha Centauri

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Aside from inventing the *warp drive*, the King clearly had a couple bad months.
There are several different endings to this story, one being that the King gives in to the inevitable, BYM reveals himself to be a charming Prince, and takes the hand of the lovely and joyful daughter. (Who would not be happy, ladies? The choice was the old toad of a count or muscular young <&rich> hero who loves you dearly.) They both lived long and prospered.

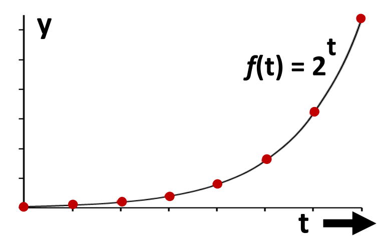
Another version tells that the King, in his shame and anger, has BYM seized and marched to the axe-man. Just as the hooded fellow raises his chopper, the original "terrible beast" bounds over the wall and gobbles the executioner, King and all the counts in waiting. Beast frees BYM and is introduced to the princess as an old friend. Same final ending. You could come up with another ending, too, but that is left as a exercise for the student.

The unconstrained growth of this story demonstrates that the coin stack starts up slowly but grows bigger at a rate that is difficult to comprehend. it is impossible to sustain this kind of growth, if for no other reason than the stack moved from Earth to 1 light year out on day 64. Physics aside, assume the workers *could* be successful at building the new required stack every single day; construction would be strangled by the need for monstrous supplies of copper for the coins and ever expanding facilities to make them. The associated logistics are so fantastic that this endeavour would have had to fail long before even day 22 (see story).

Exponential growth:

The shape:

Typical shape for base 2 growth



The process:

To get the characteristic shape,

(A) The amount of new growth next cycle will be proportional to how much is currently there.
(B) The system's growth rates should not be influenced by happenings in the external world.

This starts slowly, but the proportional growth inexorably leads to huge changes in the end.

The algebra:

Growth is in cycles per year, the quantity Q will be (such as investing at compound interest rate)

$$Q(t) = Q_0 (1+k/m)^{mt}$$

t is the elapsed time (years, seconds, etc)
Q_0 is quantity at start (t = 0)
Q(t) is quantity at any time thereafter
m is the number of cycles / time
k is the growth rate (4% means k = 0.04)

Growth is continuous, the quantity Q will be (such as using up oil from a reservoir)

$$Q(t) = Q_0 b^{kt}$$

t is the elapsed time
Q_0 is quantity at start (t = 0)
Q(t) is quantity at any time thereafter
b is base constant
k growth rate $k = 1/T_b$

T_b is time for Q to grow by a *factor* of b.
that is **:** when $t = \mathbf{T_b}$, $Q(t) = b\,Q_0$

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